TRADEMARK OF INNOVATION

Delta/Wye Transforms: Behind the Scenes

When we started learning electronics, resistors were either in series or they were in parallel and we learned how to replace such combinations with their equivalent resistances, often with the aim of reducing the entire network to a single equivalent resistance as seen by the power supply. After that came circuits (Figure 1) that contained resistors that were in neither series nor parallel but that could still be reduced by carefully identifying and reducing portions of the circuit in the right order. Notice that

R_1R1

is neither in parallel nor in series with either

R_2R2

or

R_3R3

, but by combining

R_2R2

in series with

R_4R4

and combining

R_3R3

in series with

R_5R5

, we can then combine these two equivalent resistances in parallel and, finally, combine this in series with

R_1R1

to get the total resistance seen by the supply which, using Ohm's Law, will yield the total supply current.

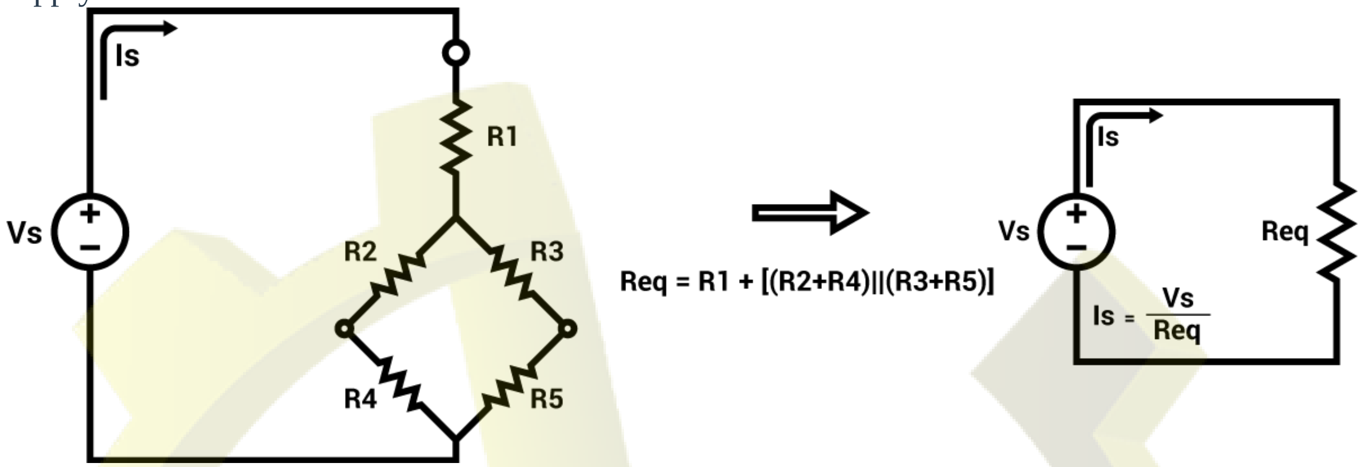

Figure 1

But now we come to circuits (Figure 2) where there aren't any pairs of resistors that are in series or in parallel -- we seem to be at an impass. One route to analyze the circuit is to fall back on Kirchhoff's Voltage Law (KCL) and Kirchhoff's Current Law (KCL) to develop a set of simultaneous algebraic equations that we can solve for the voltages and currents. While this approach will always work (for this and most kinds of circuits), it can be quite cumbersome. We might accept this as merely the cost of being able to analyze these more complex circuits, but sometimes we can avoid paying this bill by modifying, or "transforming," portions of the circuit to turn it into something that we can reduce using just the series/parallel combining rules.

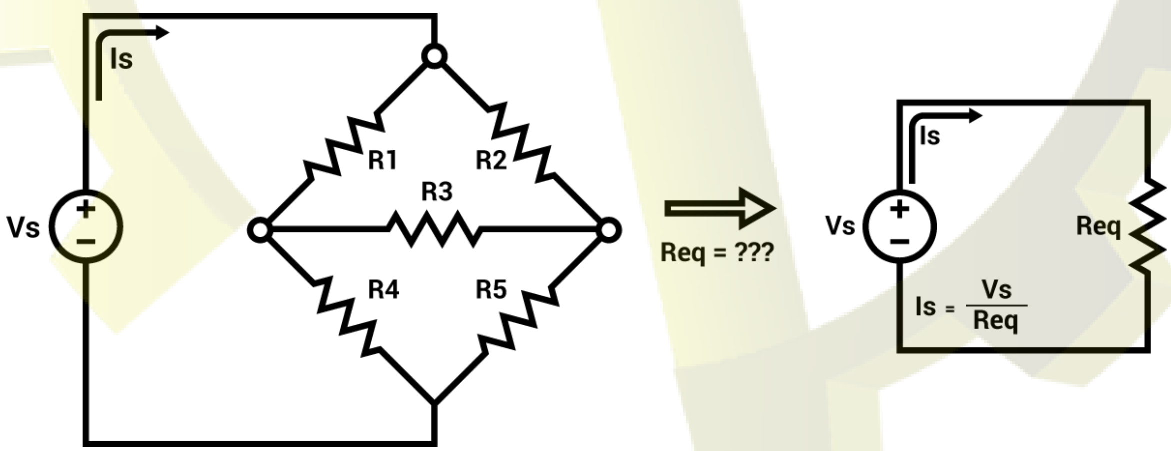

Figure 2

For simplicity, we are only going to consider DC circuits with resistors, but the principles apply to any linear system, AC or DC. Also, to keep the discussion focused, we will only find the total current supplied by the voltage source, meaning that we are looking to reduce the entire resistor network into a single equivalent resistance.

Let's look at these two circuits a bit closer in Figure 3. We see that the only difference between them is what is inside the dashed circles. In each case the circuit in the circle has three terminals that cross the circle to interact with the rest of the circuit. In the left circuit (Figure 3(a)) the resistors are connected to the terminals in a "delta" configuration (named after the capital Greek Δ) while the resistors in the right circuit (Figure 3(b)) are connected in a "wye" configuration (named after the English letter 'Y', albeit it upside down in this circuit).

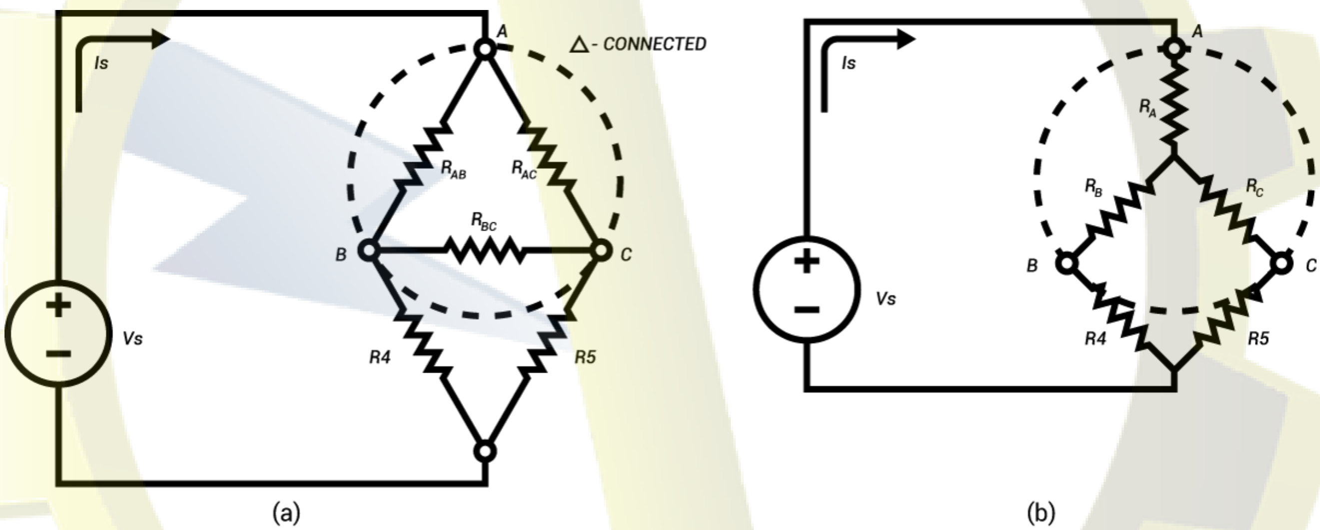

Figure 3

Now imagine putting the resistors inside the dashed circle in the left circuit into a black box, removing that box from the circuit, and replacing it with a different black box that makes the circuit behave exactly the same. Next, imagine that when you open the new box it contains three resistors arranged like those in in the right circuit. Whoever came up with the second black box very carefully chose resistor values such that the two boxes were indistinguishable to the rest of the circuit and that's the point: we know how to analyze the right circuit and we now know that when we do, the results apply to the left circuit because they are equivalent. This is the motivation behind being able to perform "delta-to-wye" and "wye-to-delta" transforms.

The Key Relationship

To determine the equations that link the resistors in the delta-connected circuit to those in the wye-connected circuit, we don't need anything beyond our trusty series/parallel forumulas (and a bit of algebra). The key is to equate the equivalent resistances between corresponding pairs of terminals while keeping the remaining terminal disconnected (Figure 4).

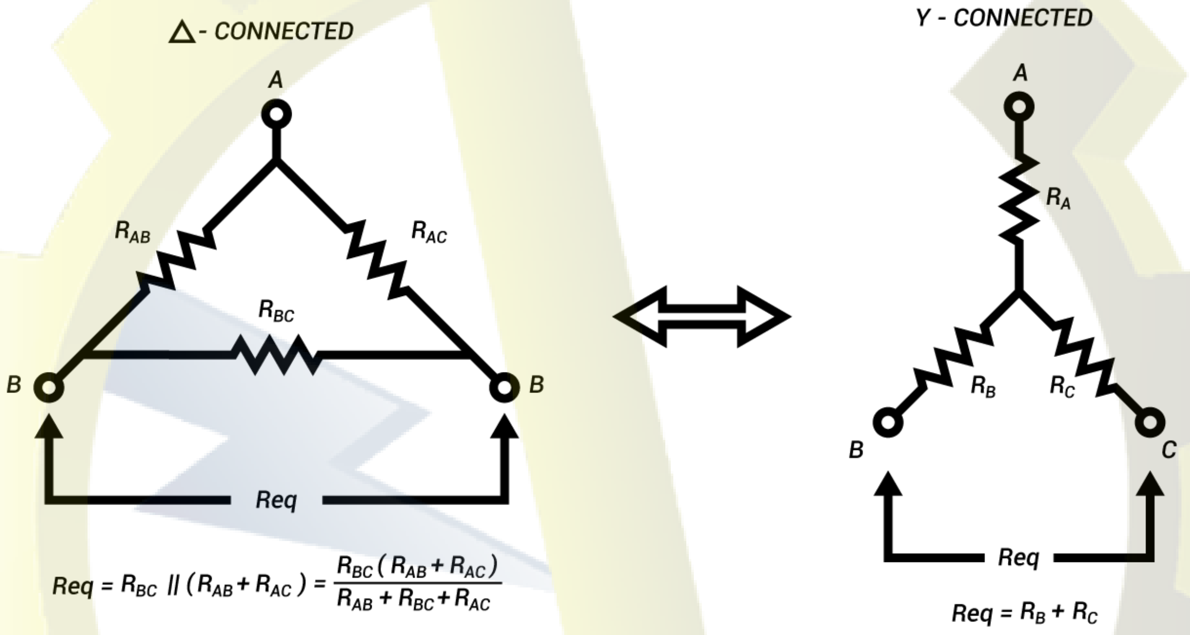

Figure 4

Doing this for the equivalent resistance seen between terminals B-C yields

$$R_B+R_C=R_{BC}(R_{AB}+R_{AC})R_{AB}+R_{BC}+R_{AC}RB+RC=RBC(RAB+RAC)RAB+RBC+RAC$$

If we repeat this process looking at the each of the other two terminal pairs in turn, we would obtain two more similar equations, but any one of them gives us the information we need (provided we recognize the symmetry involved).

Special Case: Balanced Circuits

If the resistances in each arm of a delta- or wye-connected circuit are equal, the circuit is said to be "balanced". This means that

$$R_\Delta = R_{AB} = R_{BC} = R_{AC}$$

$$R_Y = R_A = R_B = R_C$$

Combining this with the relationship from the previous section immediately yields the transform equation for the balanced case.

$$2R_Y = \frac{R_\Delta(2R_\Delta)}{3R_\Delta}$$

$$R_Y = \frac{R_\Delta}{3}$$

$$R_\Delta = 3R_Y$$

This is a much more powerful result than it might seem at first and the reason is quite simple -- when engineers design delta- or wye-connected circuits they often go to some length to make those circuits balanced. Though, of course, this is not always possible and so we need to be able to deal with the general case in which they are unbalanced.

The General Delta-to-Wye Transform

In the delta-to-wye transform we are given a known delta-connected circuit and wish to find the values for the equivalent wye-connected circuit -- hence we are trying to solve for {

$$R_A$$

,

$$R_B$$

,

$$R_C$$

} in terms of {

$$R_{AB}$$

,

$$R_{BC}$$

,

$$R_{AC}RAC$$

}.
We begin by writing our key relationship from ealier in a slighlty more compact form by defining a new quantity,

$$R_{\Delta S}R\Delta S$$

, to be the sum of all of the resistors in the delta-connected circuit.

$$R_{\Delta S}=R_{AB}+R_{BC}+R_{AC}R\Delta S=RAB+RBC+RAC$$

Next we rearrange our relation in the form of a linear algebraic equation in the unknowns {

$$R_{A}RA$$

,

$$R_{B}RB$$

,

$$R_{C}RC$$

}.

$$(0)R_{A}+(R_{\Delta S})R_{B}+(R_{\Delta S})R_{C}=R_{AB}R_{BC}+R_{BC}R_{AC}(0)RA+(R\Delta S)RB+(R\Delta S)RC=RABRBC+RBCRAC$$

Since we have three unknowns we need two more equations. These come from equating the equivalent resistances seen looking into the other two pairs of terminals. Doing so (or exploiting symmetry) we get

$$(R_{\Delta S})R_{A}+(0)R_{B}+(R_{\Delta S})R_{C}=R_{AB}R_{AC}+R_{BC}R_{AC}(R\Delta S)RA+(0)RB+(R\Delta S)RC=RABRAC+RBCRAC$$

$$(R_{\Delta S})R_{A}+(R_{\Delta S})R_{B}+(0)R_{C}=R_{AB}R_{AC}+R_{AB}R_{BC}(R\Delta S)RA+(R\Delta S)RB+(0)RC=RABRAC+RABRBC$$

By adding these two equations together and subtracting our first one, we get

$$2(R_{\Delta S})R_{A}=2R_{AB}R_{AC}2(R\Delta S)RA=2RABRAC$$

$$R_{A}=R_{AB}R_{AC}R_{\Delta S}RA=RABRACR\Delta S$$

We can solve for the other two unknown resistances (or exploit symmetry) to get

$$R_B=R_{AB}R_{BC}R_{\Delta S}RB=RABRBCR\Delta S$$

$$R_C=R_{AC}R_{BC}R_{\Delta S}RC=RACRBCR\Delta S$$

These relationships can be summarized very compactly: The resistance connected to each node in the equivalent wye-connected circuit is equal to the product of the resistances connected to the corresponding node in the delta-connected circuit divided by the sum of all the resistors in the delta-connected circuit. This is commonly expressed in a formula such as

$$R_N=R_{N1}R_{N2}R_{\Delta S}RN=RN1RN2R\Delta S$$

where

$$R_NRN$$

is the Y-connected resistor attached to terminal N while

$$R_{N1}RN1$$

and

$$R_{N2}RN2$$

are the two

$$\Delta\Delta$$

-connected resistors attached to terminal N.

The General Wye-to-Delta Transform

In the wye-to-delta transform we are given a known wye-connected circuit and wish to find the values for the equivalent delta-connected circuit; hence, we are trying to solve for {

$$R_{AB}RAB$$

,

$$R_{BC}RBC$$

,

$$R_{AC}RAC$$

} in terms of {

$$R_ARA$$

,

R_BRB

,

R_CRC

}.

This is not as straightforward as the delta-to-wye case because the unknown resistances are multiplied together, making the resulting simultaneous equations nonlinear. Fortunately, we can sidestep this inconvenience by considering the ratio of the resistors in each circuit. For instance, taking the ratio of

R_ARA

to

R_BRB

yields

$$R_A R_B = R_{AB} R_{AC} R_{AB} R_{BC} = R_{AC} R_{BC}$$ RARB=RABRACRABRBC=RACRBC

In words, the ratio of the resistors connected to any two terminals in the wye-configuration is equal to the ratio of the resistors connecting those same two terminals to the third terminal in the delta-configuration. Hence the other two ratios are

$$R_B R_C = R_{AB} R_{AC}$$ RBRC=RABRAC

$$R_A R_C = R_{AB} R_{BC}$$ RARC=RABRBC

Armed with this, we could go back to our key relationship and work forward, but it is simpler to use one of the relationships in the general delta-to-wye transform as our starting point.

$$R_A = R_{AB} R_{AC} R_{AB} + R_{BC} + R_{AC}$$ RA=RABRACRAB+RBC+RAC

$$R_{AB} R_{AC} = R_A (R_{AB} + R_{BC} + R_{AC})$$ RABRAC=RA(RAB+RBC+RAC)

$$R_{AB} = R_A (R_{AB} + R_{BC} + R_{AC} R_{AC})$$ RAB=RA(RAB+RBC+RACRAC)

$$R_{AB} = R_A (R_{AB} R_{AC} + R_{BC} R_{AC} + 1)$$ RAB=RA(RABRAC+RBCRAC+1)

$$R_{AB} = R_A (R_B R_C + R_B R_A + 1)$$ RAB=RA(RBRC+RBRA+1)

$$R_{AB} = R_A + R_B + R_A R_B R_C$$ RAB=RA+RB+RARBRC

The other two expressions obtained similarly (or, by symmetry) are

$$R_{BC}=R_B+R_C+R_BR_CR_A$$ RBC=RB+RC+RBRCRA

$$R_{AC}=R_A+R_C+R_AR_CR_B$$ RAC=RA+RC+RARCRB

These relationships can be summarized very compactly: The resistance connected between each pair of nodes in the equivalent delta-connected circuit is equal to the sum of the two resistors connected to the corresponding nodes in the delta-connected circuit plus the product of these two resistors divided by the third.

A common way of expressing this is to put the right-hand side over a common denominator and then note that the numerator in each relation is the sum of the products of each pair of resistances in the wye-connected circuit and the denominator is the resistor connected to the third terminal.

$$R_{AB}=R_PR_C$$ RAB=RPRC

;

$$R_P=R_AR_B+R_BR_C+R_AR_C$$ RP=RARB+RBRC+RARC

Example

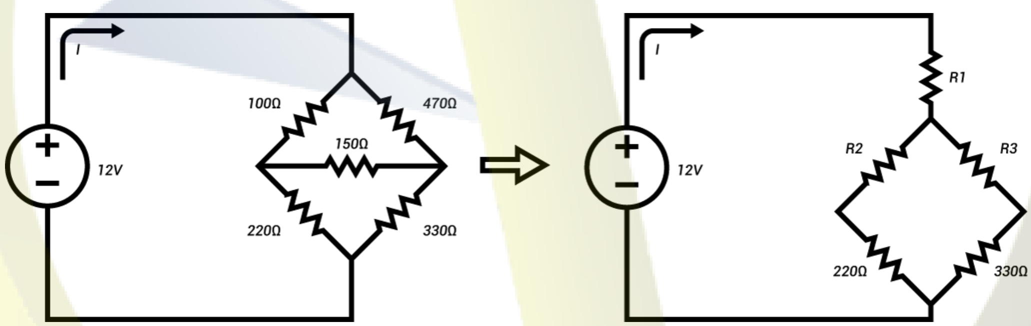

Figure 5

Let's work the problem shown in Figure 5. Before we begin, let's bound the expected answer so that we will have a good check on whether our final answer is likely correct. To do this, let's consider the role of the 150 Ω bridge resistor. The resistor serves to reduce the overall resistance by providing a path between the left side and right side of the circuit. Therefore, the highest effective resistance would occur if this resistor were removed entirely, in which case the total resistance would be the parallel combination of the left leg and the right leg, yielding

$$R_{eqmax}=(100Ω+220Ω)||(470Ω+330Ω)=228.6Ω$$ Reqmax=(100Ω+220Ω)||(470Ω+330Ω)=228.6Ω

On the other hand, the lowest overall resistance would be obtained by reducing the bridge resistor to a direct short, in which case the total resistance would be the parallel combination of the upper two resistors in series with the parallel combination of the lower two resistors, yielding

$$R_{eqmin}=(100\Omega||470\Omega)+(220\Omega||330\Omega)=214.5\Omega$$
Reqmin=(100Ω||470Ω)+(220Ω||330Ω)=214.5Ω

We therefore KNOW that our answer MUST be between these two limits. In many cases, a simple bounding analysis such as this results in an answer that is "good enough" for the purpose at hand, but let's assume that that's not the case here. Using the delta-to-wye transform equations above, we first determine the sum of the delta resistors.

$$R_{\Delta S}=100\Omega+150\Omega+470\Omega=720\Omega$$
RΔS=100Ω+150Ω+470Ω=720Ω

And then find the value of

$$R_1$$
R1

by multiplying the two resistors that branch out from the top terminal and dividing that by the sum of all three.

$$R_1=\frac{100\Omega\cdot 470\Omega}{720\Omega}=65.28\Omega$$
R1=100Ω·470Ω720Ω=65.28Ω

We next repeat this for

$$R_2$$
R2

.

$$R_2=\frac{100\Omega\cdot 150\Omega}{720\Omega}=20.83\Omega$$
R2=100Ω·150Ω720Ω=20.83Ω

We could repeat this yet again for

$$R_3$$
R3

, but let's instead use the ratiometric properties to find it.

$$\frac{R_3}{R_1}=\frac{150\Omega}{100\Omega}\Rightarrow R_3=1.5R_1=97.92\Omega$$
R3R1=150Ω100Ω⇒R3=1.5R1=97.92Ω

Now that we have all of the resistance for the equivalent wye circuit, we can determine the overall resistance very handily.

$$R_{eq}=R_1+[(R_2+220\Omega)||(R_3+330\Omega)]=219.4\Omega$$
Req=R1+[(R2+220Ω)||(R3+330Ω)]=219.4Ω

Since this is comfortably between our min and max bounds, we have high confidence that this is the correct answer or, even if we made a mistake, that our answer is quite close. The resulting total current is therefore

$$I=\frac{12V}{219.4\Omega}=54.7mA$$

$$I=\frac{12V}{219.4\Omega}=54.7mA$$

Conclusion

We have now seen that Delta/Wye transforms are useful and, more importantly, seen how they can be readily derived using nothing more than the concept of equivalent resistances using series/parallel combinations of resistors. This could serve you well since it gives you the ability to derive these relationships on the fly should the occasion ever arise and you do not have a suitable reference handy. But more importantly, this should serve to cement these fundamental concepts more firmly in the toolbox that is your mind, enabling you to wield ever more effective circuit analysis skills in your work.

Before closing we should take note of a common misconception, which is that delta-wye transforms are the ONLY way to analyze circuits that cannot be reduced otherwise. In actuality, while these transforms can make our lives easier, they are not required since ANY circuit that can be analyzed with their aid can also be analyzed through the application of KVL and KCL, either directly or via one of the more formalized techniques for their application including mesh current analysis or node voltage analysis, as well as with techinues such as Thevenin equivalent circuits.